4.8 – Row Space, Column Space, and Null Space

Definition: A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ and an associated nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ are **corresponding linear systems**.

#28 Find a general solution of the system, and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system.

$$\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

Definition: The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the **general solution**. A constant vector $\mathbf{x}_0$ in the solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ is a **particular solution**.

Theorem 4.8.2 If $\mathbf{x}_0$ is any solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for the null space of A, then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form $\mathbf{x} = \mathbf{x}_0 + c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_k \mathbf{v}_k$. Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Definition: For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ the vectors

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ \mathbf{r}_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ \vdots & \qquad\qquad \vdots \\ \mathbf{r}_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

in R^n that are formed from the rows of A are called the **row vectors** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m that are formed from the columns of A are called the **column vectors** of A.

Definition: If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is denoted by row (A) and is called the **row space** of A, and the subspace of R^m spanned by the column of vectors of A is denoted by col (A) and is called the **column space** of A. The solution space of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is denoted by null (A) and is called the **null space** of A.

Theorem 4.8.1 A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Theorem 4.8.3
a) Row equivalent matrices have the same row space.
b) Row equivalent matrices have the same null space.

Theorem 4.8.4 If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

#9 Find bases for the null space and row space of A.

a. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem 4.8.5 If A and B are row equivalent matrices, then:
a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

#15 Find a basis for the subspace of R^4 that is spanned by the given vectors.
$(1,\ 1,\ 0,\ 0),\ (0,\ 0,\ 1,\ 1),\ (-2,\ 0,\ 2,\ 2),\ (0,\ -3,\ 0,\ 3)$

#17 Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.
$\mathbf{v}_1 = (1,\ -1,\ 5,\ 2)$, $\mathbf{v}_2 = (-2,\ 3,\ 1,\ 0)$, $\mathbf{v}_3 = (4,\ -5,\ 9,\ 4)$, $\mathbf{v}_4 = (0,\ 4,\ 2,\ -3)$, $\mathbf{v}_5 = (-7,\ 18,\ 2,\ -8)$

Definition: An equation that expresses a column vector of a matrix A that does not contain a pivot as a linear combination of column vectors that contain pivots is a **dependency equation**.

#**18** Find a basis for the row space of A that consists entirely of row vectors of A.

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$